

Dissipative Effects in Nonlinear Klein-Gordon Dynamics

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We consider dissipation in a recently proposed nonlinear Klein-Gordon dynamics that admits soliton-like solutions of the power-law form $e_q^{i(kx - \omega t)}$, involving the q -exponential function naturally arising within the nonextensive thermostatistics [$e_q^z \equiv [1 + (1 - q)z]^{1/(1-q)}$, with $e_1^z = e^z$]. These basic solutions behave like free particles, complying, for all values of q , with the de Broglie-Einstein relations $p = \hbar k$, $E = \hbar \omega$ and satisfying a dispersion law corresponding to the relativistic energy-momentum relation $E^2 = c^2 p^2 + m^2 c^4$. The dissipative effects explored here are described by an evolution equation that can be regarded as a nonlinear version of the celebrated telegraphists equation, unifying within one single theoretical framework the nonlinear Klein-Gordon equation, a nonlinear Schroedinger equation, and the power-law diffusion (porous media) equation. The associated dynamics exhibits physically appealing soliton-like traveling solutions of the q -plane wave form with a complex frequency ω and a q -Gaussian square modulus profile.

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The spatio-temporal behavior of a wide family of physical systems and processes is described by non-linear partial differential equations [1–3]. This has stimulated an increasing research activity on the dynamics associated with a class of evolution equations that includes nonlinear versions of the Schrödinger [3–5] and the Fokker-Planck [6–10] ones. Our main concern here will be with a family of telegraphists-like equations describing dissipative effects in the context of a recently advanced nonlinear Klein-Gordon dynamics (NLKGD) [4] related to nonextensive statistical mechanics and the associated nonadditive entropies [11–13].

The free-particle nonlinear Klein-Gordon equation proposed in [4] has a nonlinearity in the mass term which, in contrast to what happens in the standard linear case, is proportional to a power of the wave function $\Phi(x, t)$. The salient feature of the NLKGD introduced in [4] is that it exhibits soliton-like localized solutions where the space-time dependence of the wave function $\Phi(x, t)$ occurs solely through the combination $x - vt$. Consequently, one has a space translation at a constant velocity v without change in the wave function's shape. These soliton-like solutions are known as q -plane waves and are compatible, for all values of q , with the Planck and de Broglie relations, satisfying $E = \hbar \omega$ and $p = \hbar k$, with $E^2 = c^2 p^2 + m^2 c^4$. It was shown in [4] that there is also a nonlinear Schroedinger equation (often referred to as the NRT Schroedinger equation) with a nonlinearity in the Laplacian term, that also admits q -plane wave solutions, which are compatible with the non relativistic relation $E = p^2/2m$. Under Galilean transformations the q -plane

wave solutions of the NRT Schroedinger equation recover the transformations rules of the linear Schrödinger equation [14]. The NRT equation satisfied by the q -plane waves can be obtained from a field theory based upon an action variational principle [15]. These properties suggest that the q -plane wave solutions of both the nonlinear NRT Schroedinger and Klein-Gordon equations studied in [4] can be regarded as a new field theoretical description of particle dynamics that may be relevant in diverse areas of physics, including nonlinear optics, superconductivity, plasma physics, and dark matter [15, 16].

As already mentioned, the dissipative nonlinear Klein-Gordon dynamics that we are going to explore here is described by a family of telegraphists-like equations. The standard telegraphists equation constitutes a cornerstone of mathematical physics, with deep theoretical significance and manifold applications [17–25]. Historically the telegraphists equation was first formulated to describe leaky electrical transmission lines. In one dimension it has the form,

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \delta \frac{\partial \Psi}{\partial t} = 0. \quad (1)$$

This equation corresponds to phenomena intermediate between wave propagation and diffusion. It can also be regarded as a wave equation with a damping effect described by the term having the first time derivative. It admits a statistical interpretation in terms of a Poisson process (dichotomous diffusion) associated with particles that move with constant speed and change the direction

of motion at random times [18–20]. It has profound (and surprising) connections with quantum mechanics, being intimately related to the Dirac equation [19]. The applications of the telegraphists equation are diverse. We can mention correlated random walks [17], tunneling processes [21], diffusion phenomena in optics [22, 23], and cosmic ray transport [24, 25].

The q -plane waves arise naturally within a theoretical framework where the Boltzmann-Gibbs (BG) entropy and statistical mechanics are generalized through the introduction of a power-law entropic functional S_q characterized by an index q (BG being recovered in the limit $q \rightarrow 1$). Recent progress along these lines of enquiry includes, for instance, nonlinear extensions of various important equations of physics and new forms of the Central Limit Theorem [26]. The q -Gaussian distributions, which generalize the standard Gaussian distribution and arise from the optimization of the q -entropy [11], or as solutions of the corresponding nonlinear Fokker-Planck equation [9], play a central role within these developments. They have found several interesting applications to the analysis of recent experimental findings [12]. These applications concern diverse physical systems including, among others, (i) cold atoms in dissipative optical lattices [27]; (ii) quasi-two dimensional dusty plasmas [28]; (iii) ions in radio frequency traps interacting with a buffer gas [29]; (iv) RKKY spin glasses, like CuMn and AuFe [30]; (v) Overdamped motion of vortices in type II superconductors [31]. More generally, q -exponential distributions have also been applied to a variegated set of physical scenarios. As recent examples we can mention the description of the transverse momentum spectra in high-energy proton-proton and proton-antiproton collisions [32], universal financial [33] and biological [34] laws, among others.

Of the three nonlinear dynamical equations admitting q -plane wave solutions advanced in [4] (the NRT Schroedinger, and the q -nonlinear Klein-Gordon and Dirac equations) the NRT Schroedinger equation is the one that has been more intensively studied so far. Recent advances along these lines are the investigation of an associated field theory [15], of the effects of Galilean transformations [14], of quasi-stationary, wave packet, and uniformly accelerated solutions [5, 35], and of its relation with the Bohmian formulation of quantum mechanics [36].

The nonlinear Klein-Gordon dynamics introduced in [4] is governed by the field equation,

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right] - \nabla^2 \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right] + q \frac{m^2 c^2}{\hbar^2} \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right]^{2q-1} = 0, \quad (2)$$

where $\vec{x} \in \mathbb{R}^d$, $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right)$ is the d -dimensional ∇ -operator, $q \geq 1$ and the real, positive constant Φ_0 leads to correct physical dimensionalities for all terms

(this scaling becomes irrelevant only in the limit case of the linear Klein-Gordon equation, that is, for $q = 1$). The constant Φ_0 constitutes a parameter characterizing the evolution equation (2) itself (that is, it should not be regarded as part of the initial conditions). The dynamical equation (2) can be obtained within a classical field theory derived from an appropriate Lagrangian variational principle [37].

The q -plane wave solutions of the field equation (2) are given by a q -exponential function evaluated on a pure imaginary argument, which corresponds to the principal value of

$$\exp_q(iu) = [1 + (1 - q)iu]^{1/(1-q)}; \exp_1(iu) \equiv \exp(iu), \quad (3)$$

where $u \in \mathbb{R}$. The basic relations satisfied by the above function are [38],

$$\begin{aligned} \exp_q(\pm iu) &= \cos_q(u) \pm i \sin_q(u), \\ \cos_q(u) &= r_q(u) \cos \left\{ \frac{1}{q-1} \arctan[(q-1)u] \right\}, \\ \sin_q(u) &= r_q(u) \sin \left\{ \frac{1}{q-1} \arctan[(q-1)u] \right\}, \\ r_q(u) &= [1 + (1 - q)^2 u^2]^{1/[2(1-q)]}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} \exp_q(iu) \exp_q(-iu) &= [r_q(u)]^2 = \exp_q(-(q-1)u^2), \\ \exp_q(iu_1) \exp_q(iu_2) &\neq \exp_q[i(u_1 + u_2)], \quad (q \neq 1). \end{aligned} \quad (5)$$

It is plain from Eqs. (4)-(5) that a q -exponential with a pure imaginary argument, $\exp_q(iu)$, exhibits an oscillatory behavior with a u -dependent amplitude $r_q(u)$. It can immediately be verified that the function $\exp_q(iu)$ is of square integrable for $1 < q < 3$, the concomitant integral being divergent both for $q \leq 1$ and $q \geq 3$.

The d -dimensional q -plane wave solution of equations (2) is given by

$$\Phi(\vec{x}, t) = \Phi_0 \exp_q \left[i(\vec{k} \cdot \vec{x} - \omega t) \right], \quad (6)$$

If we take into account that $d \exp_q(z)/dz = [\exp_q(z)]^q$ and $d^2 \exp_q(z)/dz^2 = q[\exp_q(z)]^{2q-1}$ we obtain, for the $(d+1)$ -dimensional d'Alembertian operator,

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \left(\frac{\Phi}{\Phi_0} \right) = -q \left[\left(\frac{\omega}{c} \right)^2 - \left(\sum_{n=1}^d k_n^2 \right) \right] \left(\frac{\Phi}{\Phi_0} \right)^{2q-1}. \quad (7)$$

Using the above relation it can be verified that the q -plane wave ansatz (6) satisfies the nonlinear field equations (2) if the frequency ω and the momentum k comply with the relation,

$$\omega^2 = c^2 k^2 + \frac{m^2 c^4}{\hbar^2}. \quad (8)$$

Making now, through the celebrated de Broglie and Planck relations, the identifications $\vec{k} \rightarrow \vec{p}/\hbar$ and $\omega \rightarrow E/\hbar$, it is plain that the q -plane waves are solutions of equation (2) satisfying $E^2 = c^2 p^2 + m^2 c^4$. That is, they comply with the energy spectrum of a relativistic free particle for all values of q . Therefore (2), together with its solution Eq. (6), constitute promising candidates for describing interesting types of physical phenomena.

The structure of the nonlinear Klein-Gordon equation in d -dimensional space [4] comprises two parts: a term corresponding to the linear $(d+1)$ -dimensional wave equation plus a nonlinear mass term proportional to a power of the wave function. Here we are going to introduce a family of evolution equations endowed with a more general power-law nonlinear term (that incorporates the one appearing in the nonlinear Klein-Gordon as a particular instance) that preserve the soliton-like q -plane wave solutions of the NLKGD.

Let us consider the equation of motion,

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right] - \nabla^2 \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right] + q \sum_{i=1}^L \delta_i \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right]^{\alpha_i^{(1)}} \left(\frac{\partial}{\partial t} \left[\frac{\Phi(\vec{x}, t)}{\Phi_0} \right] \right)^{\alpha_i^{(2)}} = 0, \quad (9)$$

characterized by the $(3L+1)$ parameters q , δ_i , $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$, with $i = 1, \dots, L$. As in the case of the nonlinear Klein-Gordon equation, the constant Φ_0 guaranties the correct dimensionalities of the different terms appearing in (9). The parameters q , $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$ are dimensionless, while the dimensions of the δ_i 's depend on the values of the exponents $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$. It can be verified after some algebra that the evolution equation (9) admits solutions of the q -plane wave form (from now on we adopt the notation $\Psi = \Phi/\Phi_0$),

$$\Psi = [1 + (1-q)i(\vec{k} \cdot \vec{x} - \omega t)]^{\frac{1}{1-q}}, \quad (10)$$

with $q > 1$, provided that the exponents $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$ comply with the consistency relation,

$$\alpha_i^{(1)} + q\alpha_i^{(2)} = 2q - 1, \quad i = 1, \dots, L. \quad (11)$$

and the wave number vector \vec{k} is related to the frequency ω through the dispersion relation,

$$-\frac{\omega^2}{c^2} + k^2 + \sum_{i=1}^L \delta_i (-i\omega)^{\alpha_i^{(2)}} = 0, \quad (12)$$

where $k^2 = \vec{k} \cdot \vec{k}$. The q -plane wave (10) constitutes a solution of (9) for any q . However, we shall consider only $q > 1$, yielding $\lim_{|\vec{k} \cdot \vec{x}| \rightarrow \infty} |\Psi|^2 = 0$, while for $q < 1$

one has $\lim_{|\vec{k} \cdot \vec{x}| \rightarrow \infty} |\Psi|^2 = \infty$. In the case of $L = 1$, $\delta_1 = m^2 c^2 / \hbar^2$, and $\alpha_1^{(2)} = 0$ equation (9) coincides with the nonlinear Klein-Gordon equation proposed in [4] which, in turn, reduces to the standard Klein-Gordon equation in the limit $q \rightarrow 1$. On the other hand, for $L = 1$, $\delta_1 > 0$, $\alpha_1^{(2)} = 1$, and $q \rightarrow 1$ the standard linear telegraphists equation is recovered. Other relevant equations are obtained as particular limit cases of (9). For instance, the limit $c \rightarrow \infty$ corresponds to equations respectively equivalent to the NRT nonlinear Schroedinger equation (for $L = 1$, δ_1 pure imaginary, and $\alpha_1^{(2)} = 1$) and to the porous media equation (for $L = 1$, δ_1 real, $q \neq 1$, and $\alpha_1^{(2)} = 1$).

We shall assume a wave vector \vec{k} with real components (this choice can be regarded as a choice determining the form of the initial form of the wave function at $t = 0$). The frequency of the q -plane wave solutions is then determined by solving the dispersion relation (12) for ω . In general we are going to have a complex frequency,

$$\omega = \omega_a + i\omega_b, \quad (13)$$

the imaginary part ω_b corresponding to the dissipation effects implied by the evolution equation (9).

The qualitative features of the dynamics associated with the q -plane wave solutions can be clarified by considering the behavior of its squared wave function profile. We have,

$$|\Psi|^2 = A [1 - (1-q)(\vec{x} - \vec{x}_0)^T \mathcal{B}(\vec{x} - \vec{x}_0)]^{\frac{1}{1-q}}, \quad (14)$$

where

$$A = [1 + (1-q)\omega_b t]^{\frac{2}{1-q}} \equiv (e_q^{\omega_b t})^2, \quad (15)$$

$$\vec{x}_0 = \frac{\omega_a t}{k^2} \vec{k}, \quad (16)$$

and \mathcal{B} is an $L \times L$ matrix with elements

$$\beta_{ij} = \frac{(q-1)k_i k_j}{[1 + (1-q)\omega_b t]^2}. \quad (17)$$

In (14), following a standard notational convention, $(\vec{x} - \vec{x}_0)$ is to be understood as a column vector, while $(\vec{x} - \vec{x}_0)^T$ stands for the concomitant row vector.

We see that the squared modulus profile $|\Psi|^2$ has the shape of a multi-valuated q -Gaussian with both its center \vec{x}_0 and amplitude A being time dependent. The center \vec{x}_0 (where $|\Psi|^2$ adopts its maximum value A) moves uniformly with a constant velocity $d\vec{x}_0/dt = \vec{k}\omega_a/k^2$. In

the dissipative case, corresponding to $\omega_b < 0$ (remember that we are considering $q > 1$), the amplitude A decreases according to a q -exponential law. That is, we have, $A = \exp_q^2(\omega_b t)$. Equations (10) and (14) are written with respect to an arbitrary cartesian spatial reference frame. If we choose a reference frame oriented in such a way that the x_1 axis points in the direction of the wave vector \vec{k} , then Ψ becomes independent of the remaining $(d-1)$ spatial coordinates, and $|\Psi|^2$ can be expressed in terms of one single coordinate $x = x_1$,

$$|\Psi|^2 = A [1 - (1-q)\beta(x-x_0)^2]^{\frac{1}{1-q}} \equiv Ae_q^{-\beta(x-x_0)^2}, \quad (18)$$

with A given by (15) and

$$\beta = \frac{(q-1)k^2}{[1 + (1-q)\omega_b t]^2}. \quad (19)$$

We see that in the dissipative case we have $\beta \rightarrow 0$ when $t \rightarrow \infty$. That is, the q -plane wave solution becomes less localized as it evolves. From now on, we are going to work with a reference frame oriented as explained above, so that we are going to consider an effective one dimensional problem

An interesting special case is given by $L = 2$, $\alpha_1^{(1)} = 2q - 1$, $\alpha_1^{(2)} = 0$, $\delta_1 = m^2 c^2 / \hbar^2$, $\alpha_2^{(1)} = -1$, $\alpha_2^{(2)} = 2$, and $\delta_2 = \delta > 0$. This case yields a non-dissipative, time reversible dynamics. The dispersion relation is,

$$-\left(\frac{1}{c^2} + \delta\right)\omega^2 + k^2 + \frac{m^2 c^2}{\hbar^2} = 0. \quad (20)$$

This dispersion relation is consistent with the relativistic energy-momentum relation with an effective velocity of light,

$$c^* = \frac{c}{\sqrt{1 + \delta c^2}} < c, \quad (21)$$

and an effective mass,

$$m^* = m\sqrt{1 + \delta c^2} \geq m. \quad (22)$$

For zero rest mass ($m = 0$) we obtain the evolution equation,

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \bar{\delta} \frac{1}{\Psi} \left(\frac{\partial \Psi}{\partial t} \right)^2 = 0, \quad (23)$$

where, $\bar{\delta} \equiv q\delta$. This notation stresses the fact that the structure of the above equation is q -independent. This equation has the remarkable property of admitting q -plane wave soliton-like solutions that propagate without

changing shape and with constant velocity, *for all values* $q > 1$. In contrast to what happens with the standard linear wave equation, these q -plane waves are the only traveling solutions of the form $f(kx - \omega t)$ admitted by (23). The effective velocity of these solutions is q -dependent and given by,

$$c_q = \frac{c}{\sqrt{1 + \frac{\bar{\delta}}{q} c^2}}. \quad (24)$$

We have explored dissipation effects in the nonlinear Klein-Gordon field theory recently introduced in [4]. These effects are described by a parameterized evolution equation yielding nonlinear versions of the celebrated telegraphists equation. This equation incorporates as particular instances various nonlinear evolution equations that are receiving increasing attention recently, such as the power-law diffusion equation (porous media equation) and the NRT nonlinear Schroedinger and Klein-Gordon equations (the last one corresponding to the NLKGD). The linear Klein-Gordon and telegraphists equations are also recovered as particular limit cases. The nonlinear telegraphists equation advanced here may be useful for describing a variety of physical systems or processes such as, for example, wave guides and electrical transmission lines with nonlinear amplitude-depending dissipation, and nonlinear non-Poissonian dichotomous diffusion processes.

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